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A BOUNDARY COLLOCATION METHOD FOR THE SOLUTION OF A FLOW PROBLEM IN A COMPLEX THREE-DIMENSIONAL POROUS MEDIUM

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SUMMARY

A flow problem in a complex three-dimensional domain with a free surface and mixed-type boundary conditions is solved by the boundary collocation method. The solution is expressed as a combination of source functions distributed all around the domain close to the boundary, plus a special basis function to take care of a corner singularity. The resulting procedure is compared with the boundary integral elements method and is found to be simpler and more flexible to implement and faster to compute.

INTRODUCTION

This paper deals with the problem of water flow in porous media. The problem is described in detail by Tal and Dagan,[1,2] and we describe here only its main features: A system of parallel tunnels is excavated in a saturated rock of low permeability for storing oil or oil products (Figure 1). In order to contain the product, a permanent water inflow has to be maintained. This inward flow on the tunnel's surface serves as a seal which prevents the stored products from escaping into the rock. The water is drained and evacuated from the bottom of the galleries to maintain a constant product level. In the absence of sufficient natural water recharge, this inward flow causes a continuous drop of the water table. Since in practice one is interested in maintaining a constant inflow and a fixed water table above the gallery's roof, an artificial recharge has to be supplemented to the rock. A simple artificial recharge system is a battery of wells drilled in rows between galleries (Figure 1). Neglecting the end effects and for a large number of galleries the flow field repeats itself periodically and it is enough to solve for the flow in a basic cell, described in Figure 2. This cell is bounded by the gallery's symmetry plane $x = 0$, the midplane between galeries $x = x_L$, the plane of the wells $y = 0$, and the midplane between wells $y = y_L$.

The water head (potential) φ in a saturated homogeneous isotropic rock satisfies the Laplace equation

$$\Delta\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0, \quad (x, y, z)\in\Omega \tag{1}$$

where the flow domain Ω and the Cartesian system x, y, z are defined in Figure 2.

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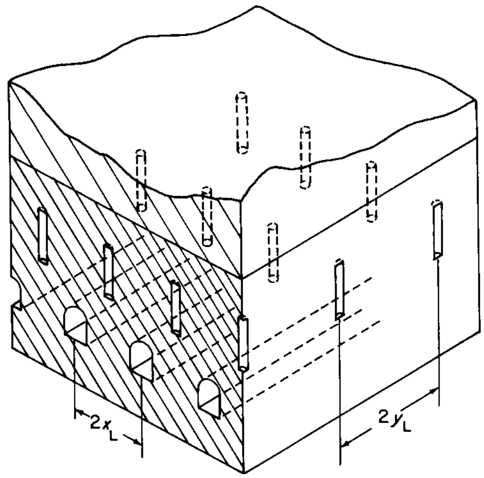

Figure 1. The system of storage tunnels and wells in a saturated rock

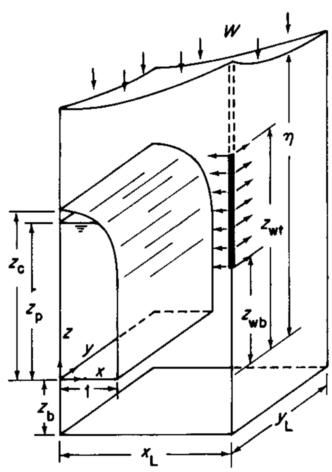

Figure 2. The basic three-dimensional cell

The boundary conditions are as follows: on the planes of symmetry $x = 0$, $x = x_L$, $y = y_L$ and on the impervious bottom $z = -z_b$:

$$\frac{\partial \varphi}{\partial n} = 0 \tag{2}$$

On the gallery surface:

$$\varphi = \varphi_g + z, \quad z \geqslant z_p \tag{3}$$

$$\varphi = \varphi_g + \frac{\gamma_p}{\gamma_w}(z_p - z) + z, \quad 0 \leqslant z \leqslant z_p \tag{4}$$

where γ_p and γ_w are the specific weights of the liquid product and water, respectively, and $\varphi_g =$

P_g/γ_w, where P_g is the vapour pressure of the product. On the perforated portion of the well:

$$\varphi = \varphi_w, \quad (x - x_L)^2 + y^2 = r_w^2, \quad z_{wb} \leqslant z \leqslant z_{wt} \tag{5}$$

where φ_w is the water head in the well's wall, r_w is the well's radius, z_{wb}, z_{wt} are the bottom and top edges of the perforated portion of the well (Figure 2). Finally, on the steady free surface $z = \eta(x, y)$ the two boundary conditions are

$$\frac{\partial \varphi}{\partial n} = \mathbf{W} \cdot \mathbf{n} \tag{6}$$

$$\varphi = \eta(x, y) \tag{7}$$

where $\mathbf{W} = (0, 0, -W)$ is the vertical recharge vector on the free surface and $\mathbf{n}$ is the unit inward vector at the free surface $z = \eta(x, y)$.

Once φ is determined by solving (1) with the boundary conditions (2)–(7), the dimensionless velocity on the gallery wall is given by $\partial\varphi/\partial n$, the derivative in the direction of the inward normal vector, and the water discharge into the gallery is

$$Q_c = \int_A \frac{\partial \varphi}{\partial n} \mathrm{d}A \tag{8}$$

where A is the gallery surface. In a steady state the inward discharge is balanced by the well recharge Q_w and the natural recharge:

$$Q_c = \frac{Q_w}{4} + W x_L y_L \tag{9}$$

There are two difficulties in computing the steady state φ:

(a) The free surface position $z = \eta(x, y)$ is unknown beforehand.

(b) The problem is essentially three-dimensional, with a boundary of a complex shape.

Tal and Dagan[1,2] solved the problem successfully by using the boundary integral element method (BIEM). The purpose of the present work is to introduce a simpler and faster procedure based on the boundary collocation method (BCM). The application of the BCM to such a complex three-dimensional problem is made possible by a special choice of the basis functions, as explained in the following section.

THE BOUNDARY COLLOCATION METHOD

Let Ω be a domain in R^2 or in R^3 with a boundary Γ and consider the linear boundary value problem

$$L\varphi(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega \tag{10}$$

$$B\varphi(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in \Gamma \tag{11}$$

where L and B are linear (differential) functionals. In the BCM we use a set of solutions $\{\varphi_i\}_{i=1}^N$ of the problem (10).

$$L\varphi_i(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \quad i = 1, \dots, N \tag{12}$$

to approximate the solution to the boundary value problem (10), (11). The collocation approximation is defined as

$$\bar{\varphi}(\mathbf{x}) = \sum_{i=1}^{N} \alpha_i \varphi_i(\mathbf{x}) \tag{13}$$

where the coefficients $\{\alpha_i\}$ are determined by the collocation conditions

$$B\bar{\varphi}(\mathbf{x}_j) = f(\mathbf{x}_j), \quad \mathbf{x}_j \in \Gamma, \quad j = 1, \ldots, N \tag{14}$$

The points $\{\mathbf{x}_j\}_{j=1}^N$ are termed as collocation points and the conditions (14) simply say that the approximation $\bar{\varphi}$ satisfies the boundary conditions (11) at the collocation points. Assuming that the problem (10), (11) is well posed, let φ denote its solution. Since $\bar{\varphi}$ automatically satisfies (10) the quality of the approximation $\bar{\varphi} \approx \varphi$ in Ω depends solely upon the quality of the approximation $B\bar{\varphi} \approx f$ on Γ. The Galerkin approach to this problem consists of determining the coefficients $\{\alpha_i\}$ in (13) so that $\| B\bar{\varphi} - f \|$ is minimized where $\| \cdot \|$ is some norm of functions on Γ. We choose to use the collocation approach, since (a) it is much simpler to implement, especially for three-dimensional problems, and (b) as argued by Levin,[3] there is no clear advantage to the Galerkin approach over the collocation approach.

Once the set of basis functions is chosen and a set of collocation points is fixed it only remains to solve the collocation equations (14) which can be written as

$$\sum_{i=1}^{N} \alpha_i B\varphi_i(\mathbf{x}_j), \quad j = 1, \ldots, N \tag{15}$$

In our case the problem is a three-dimensional Laplace equation with mixed Dirichlet–Neumann boundary conditions. In the next section we describe the application of the BCM to this problem using a special choice of basis functions. Our motivation for using these special basis function arises from the works of Hardy[4] and Franke[5] on surface fitting.

The relation of three-dimensional BCM to surface fitting is obvious, since we actually need to approximate a function ($f(\mathbf{x})$) on a two-dimensional domain Γ. The interpolation problem in R^2 consists of approximating a surface $f(x, y)$ from its values $f_j = f(x_j, y_j)$ at N distinct points $(x_j, y_j)_{j=1}^N$. Hardy[4] proposed to use for this purpose the set of reciprocal multiquadric basis functions

$$\varphi_i(x, y) = [(x - x_i)^2 + (y - y_i)^2 + d_i^2]^{-1/2}, \quad 1 \leqslant i \leqslant N \tag{16}$$

where the parameters d_i depend upon the distribution of data points. The approximating surface

$$\bar{\varphi}(x, y) = \sum_{i=1}^{N} \alpha_i \varphi_i(x, y) \tag{17}$$

is determined by the interpolation equations

$$\bar{\varphi}(x_j, y_j) = f_j, \quad j = 1, \ldots, N. \tag{18}$$

Recently, Micchelli[6] proved that the system[(18)] is always regular. Also, according to Franke,[5] the above method is one of the best global surface interpolation methods.

In fact Hardy's original motivation for using this set of basis functions originates from a Dirichlet problem in a half space:

$$\left.\begin{aligned} \Delta\varphi = 0, \quad z < 0 \\ \varphi(x, y, 0) = f(x, y) \end{aligned}\right\} \tag{19}$$

Given the approximation (17) over R^2 we readily have an approximate solution to the problem (19) in the half space $z < 0$ in the form

$$\bar{\varphi}(x, y, z) = \sum_{i=1}^{N} \alpha_i [(x - x_i)^2 + (y - y_i)^2 + (z - d_i)^2]^{-1/2} \tag{20}$$

since each term in this sum satisfies Laplace's equation for $z \leqslant 0$. Using the maximum principle for the Laplace equation

$$\max_{z<0} |\bar{\varphi}(x, y, z) - \varphi(x, y, z)| \leqslant \max_{R^2} |\bar{\varphi}(x, y) - f(x, y)| \tag{21}$$

i.e. the error in the approximation $\bar{\varphi}$ to the solution of (19) is bounded by the maximum error of the approximation to the surface $f(x, y)$ by Hardy's reciprocal multiquadrics.

Altogether the above method consists of solving Laplace's equation by a combination of fundamental solutions, or source functions, and this approach is of course not new. Approximations of flow fields around rockets on ships are often obtained by combinations of source functions with source points placed on symmetry axes of the body. Mathon and Johnston[7] introduced a non-linear procedure by which an optimal distribution of source points is found. However, this approach is too expensive even for two-dimensional problems, and therefore it can be used only for cases in which the flow field is not too complicated and thus can be approximated by a few source functions. Hardy's idea of distributing many source points close to the boundary makes the collocation method practical even in complex three-dimensional cases, such as the subject of this work. Actually, the idea of distributing sources close to the boundary can also be viewed as a variation of the boundary integral equation method[8] in which sources are distributed right on the boundary.

For the location of the source points in the three-dimensional case we make use of the experience gained in surface fitting by Hardy's multiquadrics. The rule of thumb is $d_i = 0{\cdot}825\Delta_i$ where Δ_i is a local average distance between data points near (x_i, y_i). In collocation terms the points $\mathbf{x}_i = (x_i, y_i, 0)$ are collocation points for the problem (19) and the parameters d_i are termed shift parameters. For a Laplace equation over a general domain Ω with a boundary Γ, we first choose N collocation points $\mathbf{x}_i \in \Gamma$. Then we use the above rule of thumb to determine the shift parameters d_i, and the source points are chosen as $\mathbf{x}_i + \mathbf{n}_i d_i$ where n_i is a unit outward normal vector to Γ at $\mathbf{x}_i$.

The collocation approximation to the solution of the boundary value problem (10), (11), with $L = \Delta$ in three-dimensions, is given by

$$\bar{\varphi}(\mathbf{x}) = \sum_{i=1}^{N} \alpha_i |\mathbf{x} - (\mathbf{x}_i + \mathbf{n}_i d_i)|^{-1} \tag{22}$$

where the α_i's are determined by the system of linear equations

$$\sum_{i=1}^{N} \alpha_i B |\mathbf{x}_j - \mathbf{x}_i - \mathbf{n}_i \mathbf{d}_i|^{-1} = f(\mathbf{x}_j), \quad j = 1, 2, \ldots, N \tag{23}$$

IMPLEMENTATION AND NUMERICAL RESULTS

Before solving our flow problem we adopt a major simplification of the boundary condition (5), following Tal and Dagan,[2] by replacing the well by a series of line sources. This approximation is highly satisfactory if the radius/length ratio (r_w/L_w) is very small, as is usually the case:

$$\varphi_w(x, y, z) = \sum_{i=1}^{M} \frac{q_i}{4\pi} \log \frac{[(x - x_w)^2 + (y - y_w)^2 + (z - z_i)^2]^{1/2} + (z - z_i)}{[(x - x_w)^2 + (y - y_w)^2 + (z - z_{i+1})^2]^{1/2} + (z - z_{i+1})} + c;$$
$$z_{wb} = z_1 < \cdots < z_i < \cdots < z_{M+1} = z_{wt} \tag{24}$$

In the present numerical experiments we use the same series of line sources used by Tal and Dagan. φ_w is singular on the line (x_w, y_w, z), $z_{wb} \leqslant z \leqslant z_{wt}$; therefore, as in Reference 2, we look for the

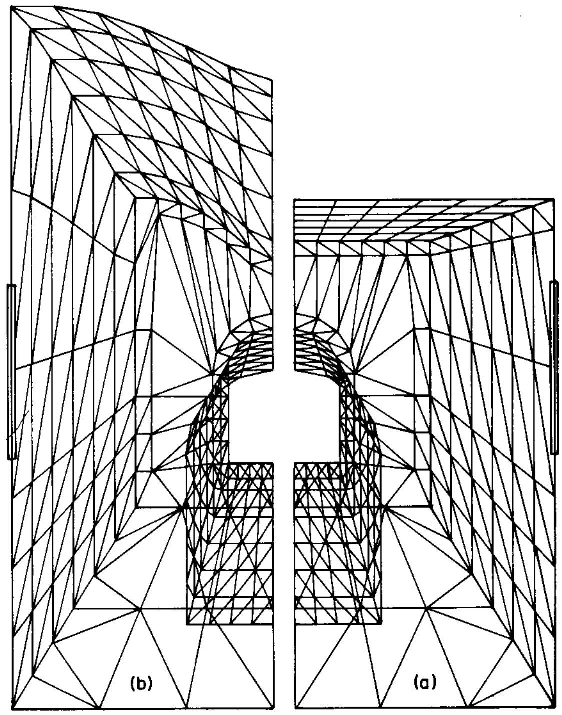

Figure 3. Boundary triangulation for the BIEM

regular part of the flow φ_r, defined by

$$\varphi(\mathbf{x}) = \sum \varphi_w(\mathbf{x}) + \varphi_r(\mathbf{x}) \tag{25}$$

where $\sum \varphi_w$ is the flow due to the system of wells.

The collocation method applied to the boundary value problem (1)–(6) can provide only an unsteady solution to the problem, since the steady free surface $\eta(x, y)$ is not known. Therefore, an iterative procedure has been adopted which starts with a horizontal free surface $\eta(x, y) = \text{constant}$; then the b.v.p. (1)–(6) is solved, and (7) is employed to calculate the new free surface $\eta(x, y)$ to be taken for the next iteration. This process continues until the relative change in η is sufficiently small.

The case represented in Figure 3 is one of a few appearing in Reference 2, and it is solved there by the BIEM. Figure 3(a) displays the initial state with a horizontal free surface and Figure 3(b) shows the steady-state shape resulting from the iterative process. The boundary triangulation is also changed from iteration to iteration, stretched up with the free surface. In applying the BCM the collocation points are initially chosen as the vertices of the triangulation used for the BIEM (207 vertices). The source points are then obtained by a normal shift of the collocation points, with a shift parameter fixed by the above-mentioned Hardy rule of thumb. At this stage the corner points of the tunnel are excluded and this may be viewed as smoothing this corner. All the other corner collocation points are common to two or three surfaces on which different Neumann-type boundary conditions are specified, and therefore they serve as multiple collocation points. Each of these points correspondingly defines two or three source points as it is shifted to diverse directions normal to the surfaces forming the corner (Figure 4). Altogether, we obtain 244 collocation points (including multiplicity) and 244 source points and thus a square system of collocation equations. Actually, each source function is taken together with its reflections w.r.t. the x- and y-axes, so that the symmetry conditions $\partial\varphi/\partial n = 0$ on $x = 0$ and on $y = 0$ are satisfied exactly.

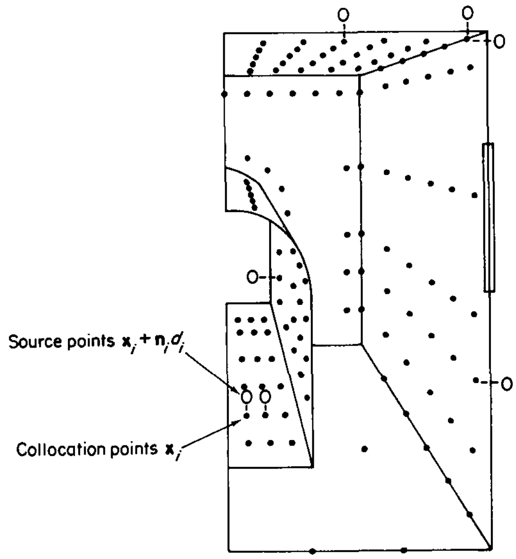

Figure 4. Collocation points and source points for the BCM

In comparing the BIEM with the BCM, we note the following:

1. The number of boundary points in the system of equations for the BCM is somewhat larger than that for the BIEM, 244 as opposed to 207.
2. The matrices of the linear systems for both methods are full and well conditioned.
3. It is much simpler to obtain the matrix for the BCM.
4. Even though the BCM system is larger than the BIEM system, the CP time for one iteration is much smaller, 40 s as opposed to 150 s on a Cyber 170–855.
5. The same number of iterations, seven, is required to achieve a stable free surface (relative change $< 0{\cdot}1$ per cent) by both methods.
6. The approximation obtained by the BCM is an easy-to-use function (22), analytic in the domain Ω and on its boundary Γ.

The last property is useful in examining the validity of the boundary conditions on a finer grid on the boundary, and that may serve us for two purposes: by comparing the appropriate boundary values of the approximation $\bar{\varphi}$ with the exact boundary values (2)–(7) on a finer grid, we obtain a good approximation to

$$e_1 = \max_{\mathbf{x}\in\Gamma_1} |\varphi(\mathbf{x}) - \bar{\varphi}(\mathbf{x})| \tag{26}$$

and

$$e_2 = \max_{\mathbf{x}\in\Gamma_2} \left| \frac{\partial\varphi}{\partial n}(\mathbf{x}) - \frac{\partial\bar{\varphi}}{\partial n}(\mathbf{x}) \right| \tag{27}$$

Here Γ_1 and Γ_2 are the boundary parts with specified Dirichlet-type and Neumann-type boundary conditions, respectively. By these error bounds for the boundary conditions and by using the maximum principle, we get an error bound for the overall error in $\bar{\varphi}$,

$$\max_{\mathbf{x}\in\Omega\cup\Gamma} |\varphi(\mathbf{x}) - \bar{\varphi}(\mathbf{x})| \leqslant e_1 + e_2 M \tag{28}$$

where $M = R^3/x_L$ (see Appendix for proof and definition of R).

The value of M may be reduced by a more careful analysis, but it depends only on the geometry of the problem. However, one can try to reduce the boundary errors e_1 and e_2 in order to gain a better global approximation. This may be done by an appropriate change of source points and collocation points.

At the first stage, we take the collocation points as the vertices of the elements used by Tal and Dagan for the BIEM, and we choose the shift parameters by Hardy's rule of thumb. Then, by evaluating the error on a finer grid on the boundary, we locate the position at which the maximum error is attained. Changing appropriately the shift parameters and the collocation points in the neighbourhood of this maximal error position, one can usually reduce e_1 and e_2. This additional effort is, of course, quite expensive and it is recommended only in cases in which there is a boundary region on which the errors are significantly prominent.

The error bound (28) gives us an error bound for the location of the free surface after convergence to steady state. The steady-state solution is also verified by checking the flow balance (9), and this is satisfied with an error of less than 1 per cent.

The accuracy of the method and its dependence upon the density of the collocation points is not yet well established. However, some indication of the order of approximation can be gathered from Franke's experiments with Hardy's method.[5] His results indicate at least an $o(h^2)$ convergence rate, where h is an average mesh size, and this rate in not too sensitive to the choice of the shift parameter d_i in (16).

In order to obtain some indication of the accuracy in our complex three-dimensional mixed boundary value problem, we performed some tests with a special set of boundary conditions for which the exact solution is known. The approximate solution developed by Tal and Dagan[2] was chosen as the known solution. It consists of a line of sources of the form (24) on the well, and two lines of sinks inside the tunnel. The various boundary conditions, computed directly from this

Table I. Velocity components at points along the tunnel's corner (upper entry—approximated value, lower entry—analytic value)

(x, y, z)	$\dfrac{\partial\varphi}{\partial x}$	$\dfrac{\partial\varphi}{\partial y}$	$\dfrac{\partial\varphi}{\partial z}$
(1, 0, 1)	0·06122	0·0	−0·13343
	0·05620	0·0	−0·13275
(1, 0·23, 1)	0·06106	0·01016	−0·13298
	0·05612	0·00955	−0·13249
(1, 0·7, 1)	0·05986	0·02997	−0·12947
	0·05487	0·02761	−0·12891
(1, 1·4, 1)	0·05602	0·05737	−0·11802
	0·05063	0·05462	−0·11547
(1, 2·33, 1)	0·04716	0·08401	−0·09119
	0·04134	0·08330	−0·08632
(1, 3·5, 1)	0·04232	0·09190	−0·05120
	0·03573	0·09134	−0·04551

approximate flow, served as inputs to the boundary collocation method. We examined the error in φ at the collocation points where $\partial\varphi/\partial n$ is given and vice versa. A typical example of the accuracy obtained is shown in Table I. These results are for the flow components at the corner of the tunnel, a part of the boundary which is most significant for the problem at hand. Thus, the fact that the test function chosen is a reasonable approximation of the actual flow, suggests that the errors for the real problem should be similar.

TREATING THE CORNER SINGULARITIES

Mathematically, we expect singular behaviour of flow velocities near the bottom corner of the galleries. Up to now, we have ignored the corner and its singularities, since from an engineering point of view the local behaviour at the corner is not important. In this section, we tackle the problem of corner singularities in order to obtain a more accurate description of the flow pattern near the corner and to check the previous results. Another object of this section is to demonstrate treatment of singularities by the collocation method in a three-dimensional domain, with the special problems of a multiply connected domain.

The singular behaviour at corners of solutions of the two-dimensional Laplace equation is well known.[9] The three-dimensional case is more complicated and there are almost no explicit formulae for the expansion of the solution near a three-dimensional corner. To find the terms in the expansion one has to solve a local eigenfunction problem.[10] In our case, the corner is essentially two-dimensional, and therefore we use the expansion known from the two-dimensional case.

We first consider the two-dimensional region described in Figure 5. According to Lehman,[9] the asymptotic expansion of the solution of a Dirichlet problem near a corner of angle $\frac{3}{2}\pi$ is

$$\varphi(x, z) - \varphi(x_c, z_c) \sim cr^{2/3} \sin \tfrac{2}{3}\vartheta + O(r)$$

as $r \to 0$ where (r, ϑ) are cylindrical co-ordinates around (x_c, z_c). We would like to add to the basis of source functions a global harmonic function with the appropriate behaviour near the corner. Any basis function should be taken together with its reflection w.r.t. the z-axis in order to satisfy the symmetry condition $\partial\varphi/\partial n = 0$ along the z-axis. Therefore, any basis function should satisfy the Laplace equation throughout the doubly connected region of Figure 5. The function $r^{2/3} \sin \frac{2}{3}\vartheta$ itself cannot be used in our case, since it can be harmonic only in a domain with a cut from (x_c, z_c) to infinity, and it is discontinuous across the cut.

Let $\zeta = x - iz$ and $\zeta_c = x_c - iz_c$ then

$$r^{2/3} \sin \tfrac{2}{3}\vartheta = \mathrm{Im}(\zeta - \zeta_c)^{2/3}$$

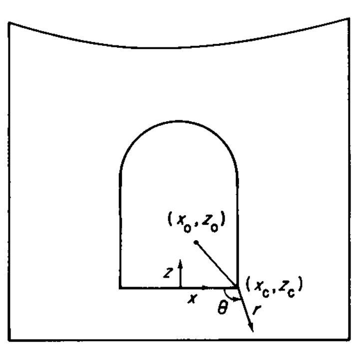

Figure 5. Co-ordinate system for the singularity treatment

The function $(\zeta - \zeta_c)^{2/3}$ is analytic in the complex plane cut from ζ_c to ∞. Let $\zeta_0 = x_0 - iz_0$ be any point in the hole of the domain, then the function

$$\left(\frac{\zeta - \zeta_c}{\zeta - \zeta_0}\right)^{2/3}$$

is analytic in the complex plane cut from ζ_c to ζ_0 and it behaves like $(\zeta - \zeta_c)^{2/3}$ as $\zeta \to \zeta_c$. Therefore, the function

$$\operatorname{Im}\left(\frac{\zeta - \zeta_c}{\zeta - \zeta_0}\right)^{2/3}$$

is harmonic in our domain and has the appropriate behaviour. We use this function, together with its reflection w.r.t. the (y, z) plane, to augment the basis of source functions for our 3-D problem. The point $(x_c, 0, 0)$ is used as an additional collocation point to complete the collocation system.

The point ζ_0 can be any point inside the cavity. However, it has been found that the results are best when the distance $|\zeta_c - \zeta_0|$ is of the same order of the smallest mesh size near the corner. In Figure 6 we exhibit the resulting normal velocities near the corner (on $y = 0$) with and without the additional singular basis function, and the difference between the two is significant. In Figure 7 we present the results with the singular basis function for two different choices of boundary collocation points. In all cases, the effect of the additional singular function was limited to the close neighbourhood of the corner.

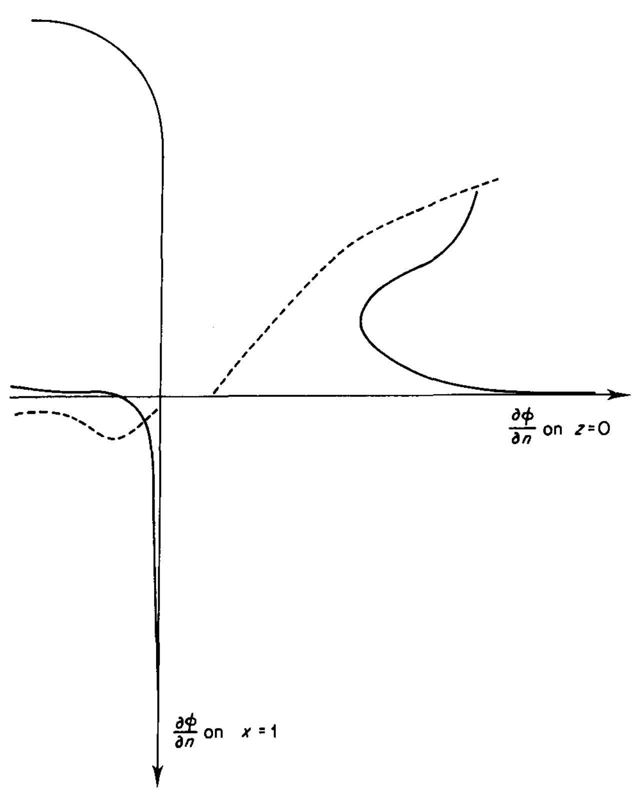

Figure 6. Normal velocities near the corner with (—) and without (---) singular term

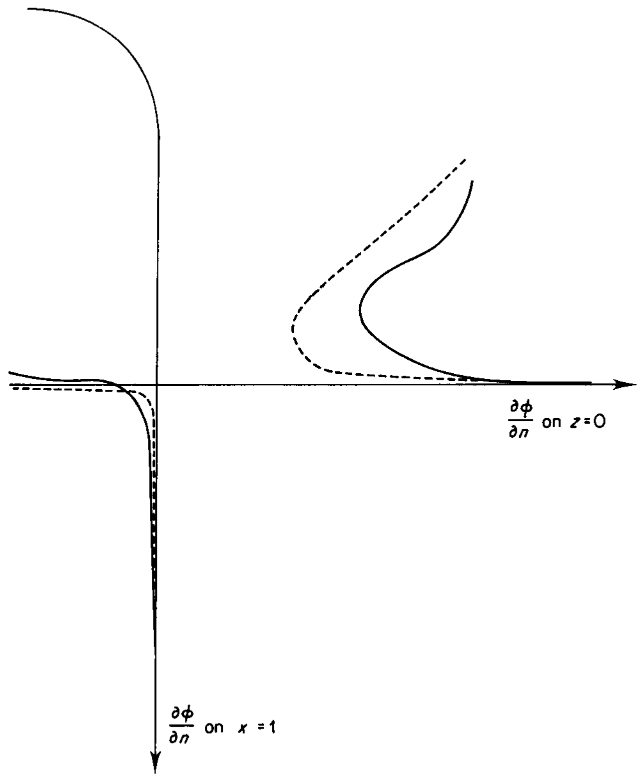

Figure 7 Normal velocities near the corner for coarse (—) and fine (---) grids with the singular term

CONCLUSIONS

In this paper the boundary collocation method has been applied to solve a flow problem in a complex three-dimensional domain with mixed-type boundary conditions. The results were compared with those of Tal and Dagan[2] using the boundary integral element method, and were found to be similar. The steady-state solution of the wells–tunnel system (Figure 3(b)), which for engineering purposes is expressed by the balance and the free surface location, turns out to be the same by the two methods. The simple global formulation of the new boundary collocation method is the reason for its main advantage, namely easy programing. Computation time is also shorter here, since the preparation of the matrices involved is faster. Local refinements are easily employed by adding collocation points, whereas in the boundary integral element method a complicated triangulation database has to be updated. As opposed to the efficiency of the boundary collocation method in the three-dimensional case, we note that collocation with fundamental solutions is not very efficient, and is not recommended for the solution of two-dimensional flow problems.

APPENDIX

Lemma

Let u satisfy $\Delta u = 0$ in a domain $\Omega \subset R^3$ and satisfy the boundary conditions

$$u \geqslant 0, \quad \text{on} \quad \Gamma_1 \tag{29}$$

and

$$\frac{\partial u}{\partial n} < 0, \quad \text{on} \quad \Gamma_2 \tag{30}$$

where $\Gamma_1 \cup \Gamma_2 = \Gamma$, the boundary of Ω, and $\Gamma_1 \cap \Gamma_2 = \varphi$. Then $u \geqslant 0$ in $\Omega \cup \Gamma$.

Proof is by the maximum principle.

Theorem

Let e satisfy $\Delta e = 0$ in the domain Ω described in Figure 2 and let

$$e_1 = \max_{\mathbf{x}\in\Gamma_1} |e(\mathbf{x})| \tag{31}$$

$$e_2 = \max_{\mathbf{x}\in\Gamma_2} \left|\frac{\partial e}{\partial n}(\mathbf{x})\right| \tag{32}$$

Then

$$|e(\mathbf{x})| \leqslant e_1 + e_2 M \tag{33}$$

where $M = R^3/x_{\mathrm{L}}$ with $R = \max_{\mathbf{x}\in\Gamma} |\mathbf{x} - (0, 0, x_{\mathrm{L}})|$.

Proof. Let h be the harmonic function

$$h(\mathbf{x}) = -A|\mathbf{x} - (\varepsilon, \varepsilon, x_{\mathrm{L}})|^{-1} + C \tag{34}$$

For an arbitrary small ε we can choose A and C such that $h \geqslant e_1$ on Γ_1 and $\partial h/\partial n < -e_2$ on Γ_2. Then $u = h - e$ satisfies the conditions of the Lemma, and thus $e \leqslant h$. In the same manner it can be shown that $-h \leqslant e$. Taking the appropriate A and C, (33) follows.

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